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Fundamental obstacles to self-pulsations in low-intensity lasers

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Abstract

We investigate most general properties of possible laser equations in the case where optics is linear. Exploiting the presence of a natural small parameter (the ratio of the photon lifetime in the laser device to the relaxation time of the population density) we establish the existence of an exponentially attracting invariant manifold which contains all bounded orbits, and show that only a small number of electromagnetic modes is sufficient to describe accurately the dynamics of the system. We give a general form of the reduced few-mode systems. We analyze the behavior of single-mode models and a double-mode model with a single optical frequency. We show that in the case where only one electromagnetic mode is excited, the rate equations are close to integrable ones, so the dynamics in this case can be understood by analytic means (by averaging method). In particular, it is shown that a non-stationary (periodic) output is possible only in relatively small (of order of some fractional powers of the small parameter) regions in the space of parameters of the system near some specially chosen parameter constellations. Estimates on the size of these regions and on the frequency of periodic self-pulsations are given for different situations.

1 Abstract laser equations. Electromagnetic modes reduction

We consider the following system of equations:

$$\begin{aligned} \dot{E} &= H(N)E, \\ \dot{N}_i &= \varepsilon F_i(N) - E^\top G_i(N)E^* \end{aligned} \tag{1}$$

where $E \in C^p$ is a complex vector, $N = (N_1, \ldots, N_k) \in R^k$ is a real vector, ε is a small parameter, the matrices $G_i(N)$ are Hermitian.

System (1) can be viewed as an abstract laser equation in the case when optics is linear. The vector E describes the electromagnetic field within the laser: this is the vector of complex amplitudes for an appropriate system of modes. We assume that the evolution of the field is governed by linear equations (i.e. the optical power is not too large). However, the evolution of the field depends on the instant state of the medium within the laser. We describe this state by the vector (or scalar if $k = 1$) N. Typically, for semiconductor devices, N is the carrier (electron/hole) density averaged over the device, or the vector of carrier densities averaged over parts of the device or taken from point to point [1]. In the absence of the field the density relaxes to a ground state; this process is governed by the first term in the equation for $\dot{N}$ while the second term is taken proportional to the intensity of the field. Many types of lasers (see [2]) are described by equations of this particular structure (with may be different interpretations of the state of the matter variables N and different choices of the set of electromagnetic modes). The actual difference between different laser devices can thus be described by different choices of the functions F, G and H in (1).

The number p of electromagnetic modes in the model may be very large, it does not matter, but we assume, however, from the very beginning, that this number is finite. This assumption means that the modes with large wave numbers must effectively average themselves, so that their contribution to dynamics must be modelled by an addition of a noise. Indeed, the noise is naturally present in any realistic situation, and the phenomenologically defined functions F, G and H are usually known with not a very good precision, so attempting to take into account very fine details of spatial structure (i.e. the modes with large wave numbers) could often be unreasonable. In fact, this paper arose from the attempt to qualitatively understand various dynamical phenomena in the multi-section distributed-feedback laser [3, 4, 5, 6, 7] which is modelled by a system of PDEs whose Galerkin or finite-element approximations fit exactly to (1).

The main idea of this paper is that many important dynamical properties of (1) can be understood without actual knowing the exact functions F, G and H, based only on the assumption of the smallness of parameter ε. It is, in essence, the ratio of photon life time in the device to the relaxation time of the medium, and it is usually reasonably small indeed. Thus, in the quoted model [4] we have $\varepsilon \sim .005$ (see [8]). More examples of lasers for which the value of ε is of the same or even higher order of smallness can be found in [9] (see also further references there).

We prove that often used single-mode, or few-mode, approximations to the laser equations are indeed correct in the limit of small ε, and we give the general procedure of reduction of the number of electromagnetic modes. It occurred also possible to give a comprehensive analyze of the dynamics of single-mode approximations. We show that if only one electromagnetic mode is excited, then non-stationary signal (self-pulsation) is, generically, impossible to produce. Roughly speaking, lasers cannot generate non-stationary signals, unless some special parameter constellations are achieved. Thus, in order to get, say, a periodic output, parameters of the laser device must be carefully tuned. How careful it should be, this depends on the actual value of ε (our analysis is valid, of course, in the limit of small ε), so we also give estimates of the size of the parameter regions which correspond to the periodic self-pulsations in different single-mode models and in some double-mode model.

Let us adopt, first, a specific terminology. The equilibrium state

$$E = 0, \quad F(N) = 0$$

will be called *the off-state*. A relative (with respect to phase shift $E \mapsto Ee^{i\varphi}$) equilibrium

$$E(t) = \sqrt{S}\xi e^{i\omega t} \tag{2}$$

will be called *a stationary state*. Obviously, (2) is a solution of (1) if and only if

$$\det(H(N) - i\omega) = 0, \qquad S = \varepsilon \frac{F_1(N)}{\xi^\top G_1(N)\xi^*} = \ldots = \varepsilon \frac{F_k(N)}{\xi^\top G_k(N)\xi^*} \tag{3}$$

where ξ is the eigenvector ($\|\xi\| = 1$) of the matrix H which corresponds to the eigenvalue $i\omega$.

A relative periodic solution will be called *self-pulsation*. We will be particularly interested in the question of existence of the self-pulsations in system (1).

We will restrict our attention to the solutions of the system for which *the norm of E does not exceed significantly that for the stationary states* (as numerics and experiments show, this is quite typical in applications, see e.g. [4, 8]). By (3), this assumption means simply that the value of E is of order $\sqrt{\varepsilon}$ in dimensionless variables. Therefore, we may scale $E \mapsto \sqrt{\varepsilon}E$ and the equations (1) will recast as

$$\begin{aligned} \dot{E} &= H(N)E, \\ \dot{N}_i &= \varepsilon(F_i(N) - E^\top G_i(N)E^*) \end{aligned} \tag{4}$$

where we will be looking for the solutions with *the finite amplitude of E* at ε sufficiently small.

Let us define *the critical set $\mathcal{N}_{cri}$* as the set of values of N for which the matrix $H(N)$ has at least one eigenvalue on the imaginary axis. According to (3), every stationary state lies in the critical set.

Lemma. *Every nontrivial finite-amplitude solution of (4) stays in a small neighborhood of $\mathcal{N}_{cri}$.*

This statement becomes obvious when explained. Fix any constant K and define a finite-amplitude solution as such for which $\|E(t)\| \leq K$ for all t; the solution is nontrivial when $E(t)$ is not identically zero. Since every nontrivial finite-amplitude solution of (4) at $\varepsilon = 0$ lies in $\mathcal{N}_{cri}$, it follows by continuity that at $\varepsilon \neq 0$ all nontrivial finite-amplitude solutions lie in a $\delta(\varepsilon)$-neighborhood of $\mathcal{N}_{cri}$, where $\delta \to 0$ as $\epsilon \to 0$ for any fixed K.

So, we have to focus on a small neighborhood of the critical set. By obvious stability reasons we can further restrict our considerations to a small neighborhood of the so-called *threshold set $\mathcal{N}_{thr}$* which is the subset of $\mathcal{N}_{cri}$ for which the matrix $H(N)$ has no eigenvalues with positive real parts.

Let us take a compact connected subset $\mathcal{N}^\circ$ of the threshold set such that at every point of $\mathcal{N}^\circ$ the matrix H has the same number m of eigenvalues on the imaginary axis (accounted with multiplicities). Generically, $\mathcal{N}^\circ$ is a smooth manifold. When N belongs to a small neighborhood of $\mathcal{N}^\circ$, the space C^p of the E-variables is decomposed into direct sum of two invariant subspaces, $\mathcal{E}_c$ and $\mathcal{E}_s$, of the matrix $H(N)$. The space $\mathcal{E}_c$ corresponds to the m eigenvalues close to the imaginary axis and $\mathcal{E}_s$ corresponds to the rest of eigenvalues which stay bounded away from the imaginary axis; both the subspaces depend smoothly on N.

We may choose a basis $\{\xi_1(N),\ldots,\xi_m(N)\}$ in $\mathcal{E}_c$ and a basis $\{\eta_1(N),\ldots,\eta_{p-m}(N)\}$ in $\mathcal{E}_s$. This gives us the following decomposition

$$E = \xi(N)U + \eta(N)V \tag{5}$$

where $\xi(N)$ is the matrix with columns $(\xi_1(N),\ldots,\xi_m(N))$ and $\eta(N)$ is the matrix with columns $(\eta_1(N),\ldots,\eta_{p-m}(N))$; thus, $U \in C^m$ and $V \in C^{p-m}$ are coordinates in $\mathcal{E}_c$ and $\mathcal{E}_s$ respectively. By construction,

$$H(N)\xi(N) = \xi(N)A(N) \tag{6}$$

and

$$H(N)\eta(N) = \eta(N)B(N) \tag{7}$$

where the spectrum of $A(N)$ lies close to the imaginary axis (it lies exactly on the imaginary axis when $N \in \mathcal{N}^\circ$) and the spectrum of $B(N)$ is bounded away from it (see Fig.1).

Plugging (5)-(7) in (4) we obtain the following system

$$\begin{aligned}
\dot{V} &= B(N)V - (\eta^\dagger(N))^\top \eta'(N)\dot{N}\, V - (\eta^\dagger(N))^\top \xi'(N)\dot{N}\, U,\\
\dot{U} &= A(N)U - (\xi^\dagger(N))^\top \xi'(N)\dot{N}\, U - (\xi^\dagger(N))^\top \eta'(N)\dot{N}\, V,\\
\dot{N}_i &= \varepsilon(F_i(N) - U^\top\xi^\top(N)G_i(N)\xi^*(N)U^* - 2\mathrm{Re}\left(U^\top\xi^\top(N)G_i(N)\eta^*(N)V^*\right) + O(\|V\|^2))
\end{aligned} \tag{8}$$

where the matrices $\xi^\dagger(N)$ and $\eta^\dagger(N)$, normed so that $(\eta^\dagger(N))^\top\eta(N) = 1$, $(\xi^\dagger(N))^\top\xi(N) = 1$, are found from the equations

$$H^\dagger(N)\xi^\dagger(N) = \xi^\dagger(N)\tilde{A}(N) \tag{9}$$

and

$$H^\dagger(N)\eta^\dagger(N)H^*(N) = \eta^\dagger(N)\tilde{B}(N), \tag{10}$$

where $H^\dagger$ denotes the matrix conjugate to H and $\tilde{A}$ and $\tilde{B}$ denote some matrices similar to those conjugate to A and B, respectively.

At $\varepsilon = 0$ the system takes the form

$$\dot{V} = B(N)V, \qquad \dot{U} = A(N)U, \qquad \dot{N} = 0. \tag{11}$$

The invariant manifold $V = 0$ of this system is exponentially asymptotically stable at N close to $\mathcal{N}^\circ$ (because the spectrum of $B(N)$ lies strictly to the left of the

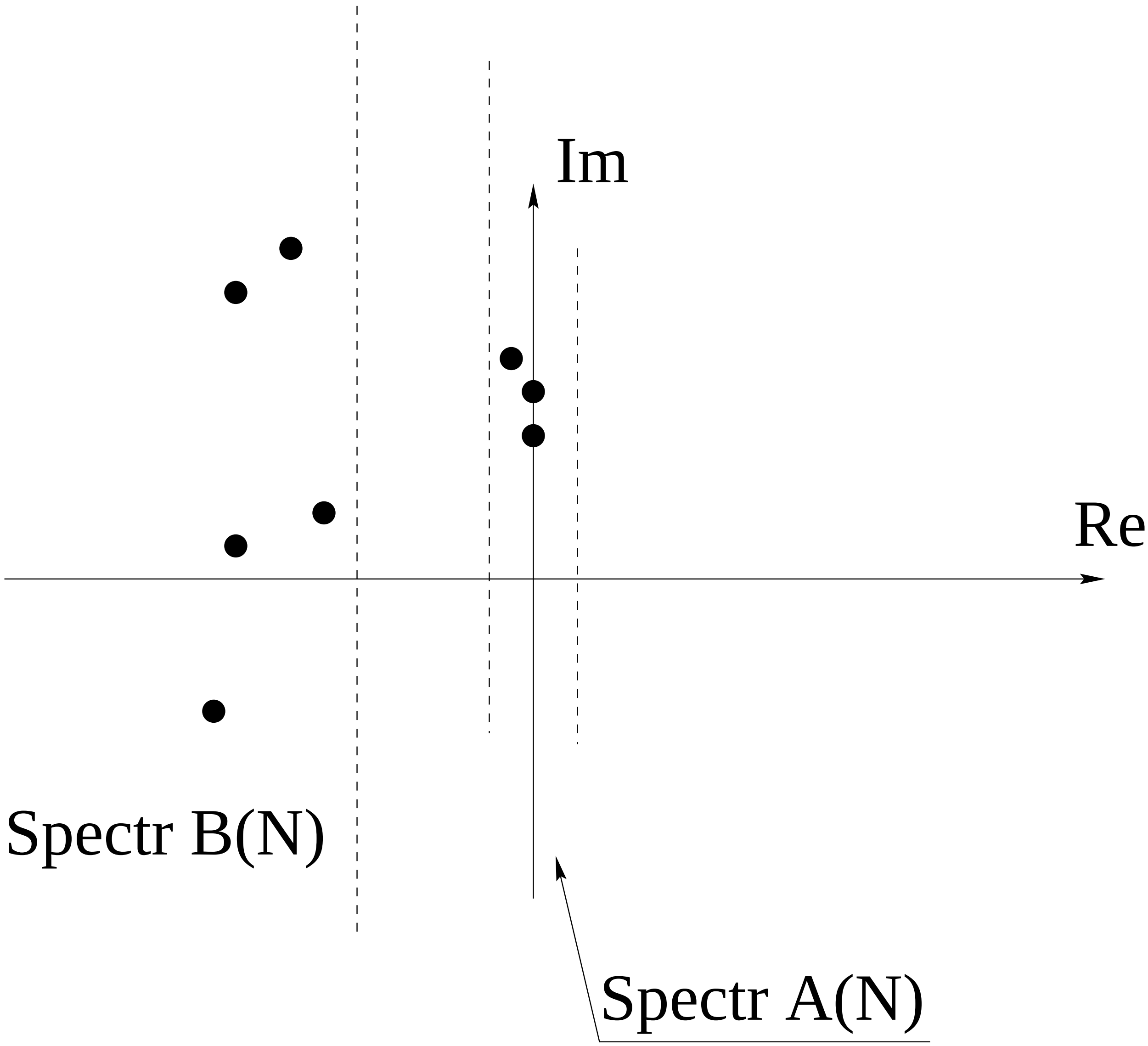


Figure 1: The spectrum of $A(N)$ lies close to the imaginary axis for all N close to the piece $\mathcal{N}^{\circ}$ of the threshold surface, and the spectrum of $B(N)$ lies strictly farther to the left from the imaginary axis.

spectrum of $A(N)$ for every fixed N under consideration). Although this manifold is not compact, it is obvious that the system (8) can be modified at $\|U\| \geq K$ (for any arbitrarily large, aforehand given K) and at N far from $\mathcal{N}^\circ$ in such a way that this manifold would become outflowing at $\varepsilon = 0$. Thus, the standard theory is applied [10] which guarantees the continuation of this invariant manifold onto nonzero ε. We formulate this as follows.

Theorem. *The system (8) has an exponentially attracting invariant smooth manifold*

$$V = \varepsilon \mathcal{V}(N, U, \varepsilon) U \tag{12}$$

where the function $\mathcal{V}$ is defined for all N in a small (independent of ε) neighborhood of $\mathcal{N}^\circ$, for all U whose norm is less than some aforehand given K (which can be taken arbitrarily large) and for all small ε (the range of ε depends on the choice of K).

Note that the invariant manifold (12) is symmetric with respect to the phase shift $(U, V) \mapsto (U, V)e^{i\varphi}$ (because system (8) is symmetric), therefore the function $\mathcal{V}$ must be invariant with respect to the rotations $U \mapsto Ue^{i\varphi}$.

It should be mentioned that since we assume our vectors E and N finite-dimensional, the proof of the theorem is obtained simply by reference to a general result of [10]. However, the result still holds true in the case where E is infinite-dimensional (belongs to a complex Hilbert space), as it is shown in [11].

According to this theorem, we may restrict our attention to the manifold (12) only (for any initial condition a trajectory must exponentially fast come to an arbitrarily small neighborhood of this manifold and stay there forever.) Plugging (12) into (8) we arrive to the following system on the invariant manifold:

$$\dot{U} = A(N)U - \alpha(N)\dot{N}\, U + O(\varepsilon^2)U,$$

$$\dot{N}_i = \varepsilon(F_i(N) - U^\top(g_i(N) + \varepsilon\tilde{g}_i(N, U))U^* + O(\varepsilon^2))$$

where

$$\alpha(N) = \xi^\dagger(N)\xi'(N), \qquad g_i(N) = \xi^\top(N)G_i(N)\xi^*(N)$$

and

$$\tilde{g}_i(N, U) = 2\mathrm{Re}\left(\xi^\top(N)G_i(N)\eta^*(N)\mathcal{V}^*(N, U, 0)\right).$$

Note that $\tilde{g}_i$ must be rotationally invariant, i.e.

$$\tilde{g}_i(N, Ue^{i\varphi}) \equiv \tilde{g}_i(N, U). \tag{13}$$

We will drop the $O(\varepsilon^2)$-terms from now on (simply because they are too small) and proceed to the study of the shortened system

$$\begin{aligned} \dot{U} &= A(N)U - \alpha(N)\dot{N}\, U, \\ \dot{N}_i &= \varepsilon(F_i(N) - U^\top(g_i(N) + \varepsilon\tilde{g}_i(N, U))U^*). \end{aligned} \tag{14}$$

System (14) can be viewed as a general form of finite-mode approximations of laser equations. We stress that we arrived to (14) from the original system (1) using only the assumption of smallness of ε (while all the other coefficients are assumed to be bounded) and the smallness (finiteness in the rescaled variables) of the amplitude of E. Making specific assumptions on the spectrum of matrix $A(N)$ we can further transform the equations, and this will even allow for a complete analysis of dynamics in some basic cases.

Namely, if only a single mode is on the threshold (i.e. $U \in C^1$ and $A(N)$ is a scalar) the system, after an appropriate rescaling of time and the N-variables, becomes close to an integrable one, so the averaging methods are very well applied here (see Secs. 2-4). For multi-mode models, the near-integrability does not always hold. However, the methods of bifurcation theory (normal forms and blow-up) can still be applied, as we demonstrate in Sec.5 for the example of a double mode on the threshold.

The overall idea of this paper is that the presence of the explicit small parameter in equations (14) allows one always to find, by expansion in (fractional) powers of ε, an appropriate coordinate transformation which would bring the equations to some normal form, mostly independent on the particular choice of the functions α, F, g.

We derive such normal forms (formulas (21), (32) and (34), (47) and (51), (57), (60), (68) and (69) below) for the cases where the laser generates only one optical frequency, i.e. when there is only one mode on the threshold, or if there are two modes on the threshold, then they both have the same frequency (which means that $A(N)$ has a double eigenvalue on the imaginary axis at this moment). The results of our analysis are as follows. We show (Sec.2) that if a single eigenvalue of H intersects the imaginary axis transversely as N crosses the threshold, then the region of the parameter values which correspond to the existence of fast self-pulsations is always small, of order ε. The frequency of these self-pulsations is of order $\varepsilon^{1/2}$ (note that our time unit is the time a photon spends in the device, so our small - of order fractional powers of ε - frequencies can correspond to sufficiently large frequencies in practice). In case we have a vector variable N (Sec.3), the system may have large parameter regions which correspond to slow (with the frequency of order ε) self-pulsations. In this regime, the N-variables oscillate staying on the threshold surface and the electromagnetic power changes passively, in such a way that it prevents the cross-threshold deviations of N. If the critical eigenvalue is tangent to the imaginary axis as N crosses the threshold (Sec.4), then the region of the existence of fast self-pulsations (with the frequency of order $\varepsilon^{2/3}$) is larger: $O(\varepsilon^{1/3}) \times O(\varepsilon^{2/3})$ (we have two parameters here which scale differently). In the case of the cubic tangency we have even larger existence region of size $O(\varepsilon^{1/4}) \times O(\varepsilon^{1/2}) \times O(\varepsilon^{3/4})$, whereas the frequency of the self-pulsations is lower, of order $\varepsilon^{3/4}$. In the case where a double eigenvalue intersects the imaginary axis (Sec.5) the region of the existence of self-pulsations has the same size as in the case of the quadratic tangency to the imaginary axis: $O(\varepsilon^{1/3}) \times O(\varepsilon^{2/3})$, but the frequency here is relatively higher - $O(\varepsilon^{1/3})$.

2 Basic single-mode model

The first case we consider is when $A(N)$ is just a scalar (i.e. we have only one pure imaginary eigenvalue on the threshold). This is, obviously, the most general case. We denote the only eigenvalue of $A(N)$ as $\lambda(N)$ in this case (i.e. $\lambda(N) \equiv A(N)$ here). To start with, we assume that we have only one N and let N^0 be the threshold value, i.e. $\mathrm{Re}\lambda(N^0) = 0$. We assume that $\lambda(N)$ crosses the imaginary axis with a non-zero velocity when N is pushed above the threshold, i.e.

$$\mathrm{Re}\lambda'(N^0) \neq 0. \tag{15}$$

Denoting $S = |U|^2$ (recall that $U \in C^1$ in the case under consideration) we arrive at the following system of rate equations in R^2

$$\begin{aligned} \dot{S} &= 2(\mathrm{Re}\lambda(N) - \mathrm{Re}\,\alpha(N)\dot{N})S, \\ \dot{N} &= \varepsilon(F(N) - (g(N) + \varepsilon\tilde{g}(N,\sqrt{S}))S). \end{aligned} \tag{16}$$

We assume that $g(N^0) \neq 0$ (for the presence of electromagnetic field must have an effect on the evolution of the matter, i.e. on the N-variable). Also, let $F(N^0) \neq 0$ (i.e. the off-state is not at the threshold). Since N must stay close to N^0 we have that $g(N) \neq 0$ and $F(N) \neq 0$ in the interesting region. Moreover, we assume that $g(N) > 0$ because it can always be achieved by a proper choice of the sign of N.

Let us now change the variables $S \mapsto S_{new}|F(N)|/(g(N) + \varepsilon\tilde{g}(N,\sqrt{S}))$ which will bring the system to the form

$$\begin{aligned} \dot{S} &= (2\mathrm{Re}\lambda(N)(1 - \varepsilon\hat{g}(N,S)) - \tilde{\alpha}(N)\dot{N} + O(\varepsilon^2))S, \\ \dot{N} &= \varepsilon|F(N)|(\pm 1 - S) \end{aligned} \tag{17}$$

where $\tilde{\alpha}(N) = 2\mathrm{Re}\,\alpha(N) + \frac{d}{dN}(\ln|F(N)|/g(N))$ and $\hat{g}(N,S) = \frac{S}{g(N)}\frac{d}{dS}\tilde{g}(N,\sqrt{S|F(N)|/g(N)})$. We may now also scale the time to $|F(N)|$ and write the system as

$$\begin{aligned} \dot{S} &= (2\mu(N)(1 - \varepsilon\hat{g}(N,S)) - \gamma(N)\dot{N} + O(\varepsilon^2))S, \\ \dot{N} &= \varepsilon(\pm 1 - S) \end{aligned} \tag{18}$$

where $\pm$ stands for the sign of $F(N)$ and $\mu(N) = \mathrm{Re}\lambda(N)/|F(N)|$, $\gamma(N) = \tilde{\alpha}(N)/|F(N)|$.

Since N has to be close to the threshold, we can write

$$N = N^0 + \delta n$$

for some small δ whose dependence on ε is to be defined. We can expand

$$\mu(N) = \mu_1\delta n + \mu_2\delta^2 n^2 + \mu_3\delta^3 n^3 + \dots$$

and

$$\gamma(N) = \gamma_0 + \gamma_1 \delta n + \dots .$$

Recall that $\mu_1 \neq 0$. The system takes the form

$$\begin{aligned}
\dot{S} &= \delta(2\mu_1 n - \gamma_0 \dot{n} + 2\mu_2 \delta n^2 + 2\mu_3 \delta^2 n^3 - 2\varepsilon \mu_1 \hat{g}(N^0, S) - \delta \gamma_1 n \dot{n} + O(\delta^3, \varepsilon\delta, \varepsilon^2/\delta))S, \\
\dot{n} &= \frac{\varepsilon}{\delta}(\pm 1 - S).
\end{aligned} \tag{19}$$

It is clear now that the wise choice of the scaling factor δ is

$$\delta^2 |\mu_1| = \varepsilon.$$

Choosing δ in this way and rescaling the time we will obtain equations which have the following limit as $\varepsilon \to +0$:

$$\dot{S} = 2\sigma n S, \quad \dot{n} = (\pm 1 - S), \tag{20}$$

where $\sigma =$ sign μ_1. It is a conservative system with the first integral $h = \sigma n^2 + S \mp \ln S$. When we have a minus sign in the second equation of (20) there is no bounded trajectories at $S > 0$. If we have a plus sign, and $\sigma = -1$, the only bounded trajectory is a saddle equilibrium state at $S = 1$ (see the phase portraits in Fig.2). The analogous conclusion holds true for the system (19) as well (because it becomes δ-close to (20) after the rescaling of time). Therefore, we will focus on the "plus-plus" case (i.e. $F(N^0) > 0$, $\mu_1 > 0$).

Here, after rescaling the time to the factor δ, system (19) takes the form

$$\begin{aligned}
\dot{S} &= (2n - \sqrt{\varepsilon}(\frac{\gamma_0}{\mu_1}(1 - S) + 2\frac{\mu_2}{\mu_1} n^2) + \varepsilon(\varphi(n) + n\psi(S)) + O(\varepsilon^{3/2}))S, \\
\dot{n} &= 1 - S
\end{aligned} \tag{21}$$

where φ and ψ are some smooth functions. This system can be viewed as a slightly refined form of the simplest laser rate equations (see [12]). It is known (see [13] and references therein) that after an appropriate rescaling these rate equations become conservative at $\varepsilon = 0$. Indeed, at $\varepsilon = 0$ system (21) takes the form:

$$\dot{S} = 2nS, \quad \dot{n} = 1 - S \qquad (S > 0), \tag{22}$$

with the first integral $h = n^2 + S - \ln S$.

The orbits of (22) are closed curves surrounding the equilibrium (of center type) $O(n = 0, S = 1)$. The line $L : \{n = 0, S > 1\}$ is a cross-section: every orbit starting on L returns to it after one finite time round about the equilibrium O. The system (21) has, of course, an equilibrium $O_\varepsilon(S = 1, n = n_\varepsilon = O(\varepsilon))$ close to O and the Poincaré map on the line $\{n = n_\varepsilon, S > 1\}$ is still defined. The form of equations (21) will not change if we shift the origin in n so that to make $n_\varepsilon = 0$, therefore we

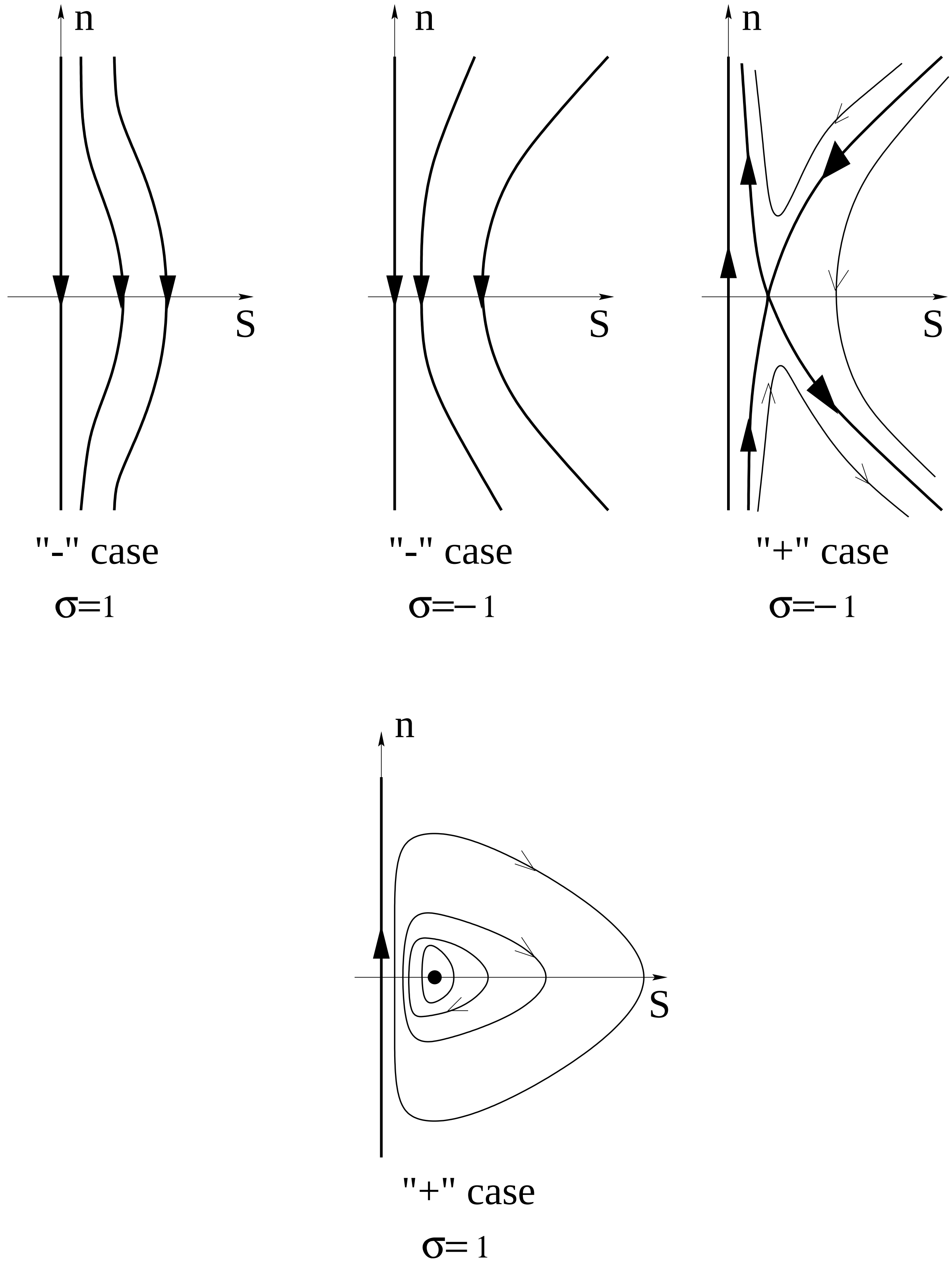


Figure 2: Phase portraits for the integrable limit (20).

will assume that $O_\varepsilon = (S = 1, n = 0)$. It means, in particular, that $\varphi(0) = 0$ in (21).

Consider now the function

$$h_\varepsilon(n,S) = n^2 + \sqrt{\varepsilon}\frac{2\mu_2}{3\mu_1}n^3 + \varepsilon\int\varphi(n)dn + S - \ln S - \frac{\varepsilon}{2}\int(1 - S^{-1})\psi(S)dS.$$

On the orbits of system (21) we have

$$\frac{d}{dt}h_\varepsilon = \sqrt{\varepsilon}\frac{\gamma_0}{\mu_1}(S-1)^2 + O(\varepsilon^{3/2})|S-1|\cdot(|n| + |S-1|)).$$

Thus, if we parametrize points on the cross-section L by the value of h_ε, the Poincaré map $h \mapsto \bar{h}$ will have the form

$$\bar{h} = h + \sqrt{\varepsilon}\frac{\gamma_0}{\mu_1}\Phi(h) + O(\varepsilon^{3/2})h \tag{23}$$

for some positive function $\Phi(h)$ such that $\Phi(h) = const \cdot h + o(h)$ at small h. It is immediately seen from (23) that the system may have a non-trivial behavior only at $\gamma_0/\mu_1 = O(\varepsilon)$. Outside of this interval the iterations of the Poincaré map (23) either converge to zero at negative γ_0 (i.e. all the orbits of system (21) converge to the stable equilibrium O), or diverge at positive γ_0 which means that the orbits of (21) leave the region of finite S (Fig.3).

The transition through $\gamma = 0$ changes the stability of O, so the Andronov-Hopf bifurcation must happen. However, as we see, the possible parameter range corresponding to the existence of the limit cycle born at this bifurcation must be of order ε. Thus, this AH-bifurcation is very sharp: when parameter γ_0/μ_1 changes, the limit cycle born from O grows in size very fast and leave the region of finite S (Fig.4). Note that this sharpness of the Andronov-Hopf bifurcation was indeed observed in different models of laser dynamics (see [9, 4]; I, personally, have learnt about this from a talk by T.Erneux, published later as [9]). The analysis given here explains from a general point of view why this type of behavior is inavoidable.

We can conclude, that
generically, in case $N \in R^1$, there can be no robust self-pulsations.

It is a disaster, of course, because it means that in order to produce self-pulsations we must create some special parameter constellations to get more modes on the threshold or to make the eigenvalues cross the imaginary axis in a non-generic way. This means that the existence of self-pulsations is sensitive to variations of parameters, so obtaining some large regions of existence of self-pulsations can not be easy, in principle.

Another bad property of system (21) is an oscillatory stability of the equilibrium state O: as it follows from (23), even when O is stable the convergence of the orbits to O is slow (the temp of convergence is of order $O(\sqrt{\varepsilon})$).

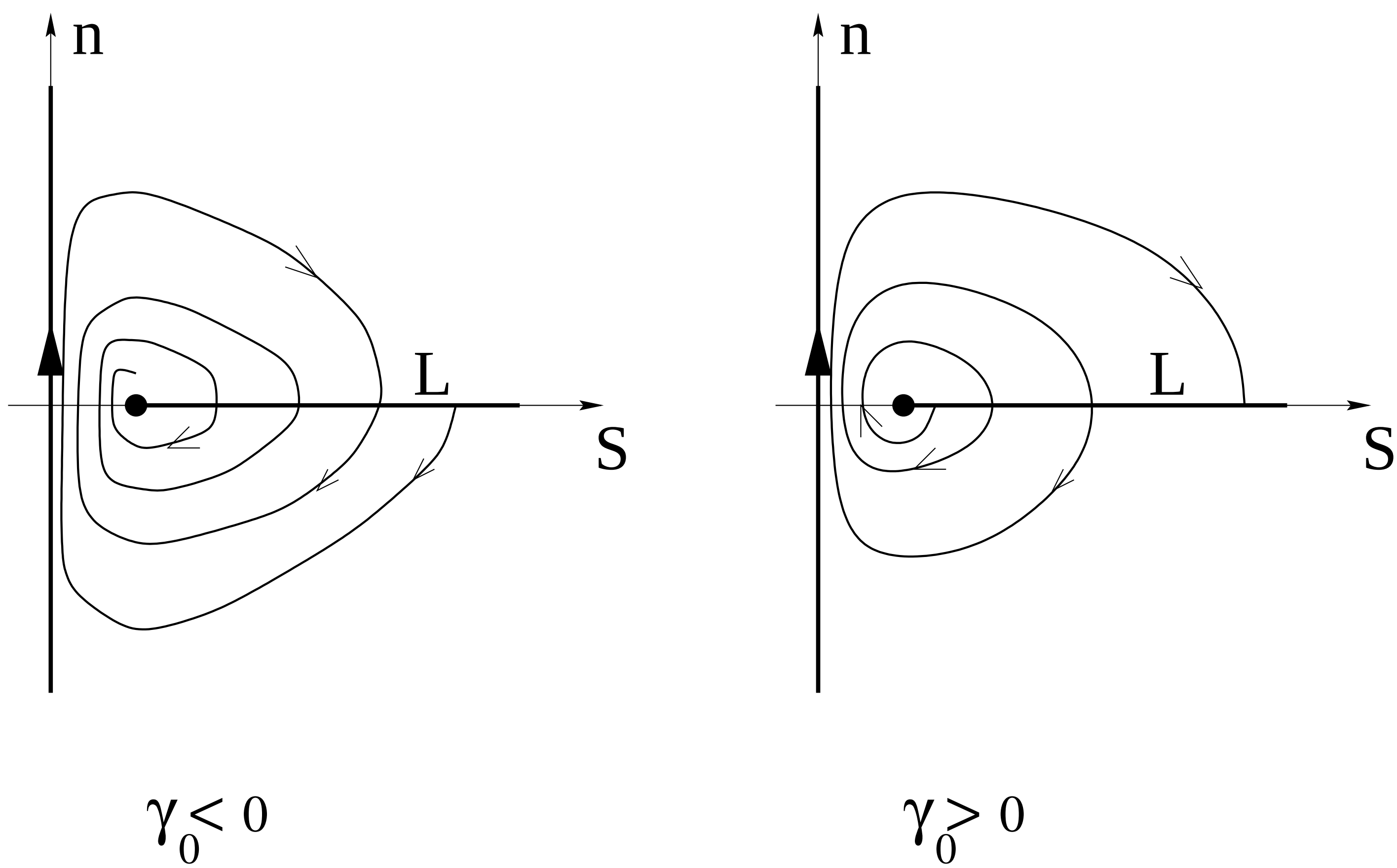


Figure 3: There are no limit cycles in (21) at γ_0 bounded away from zero.

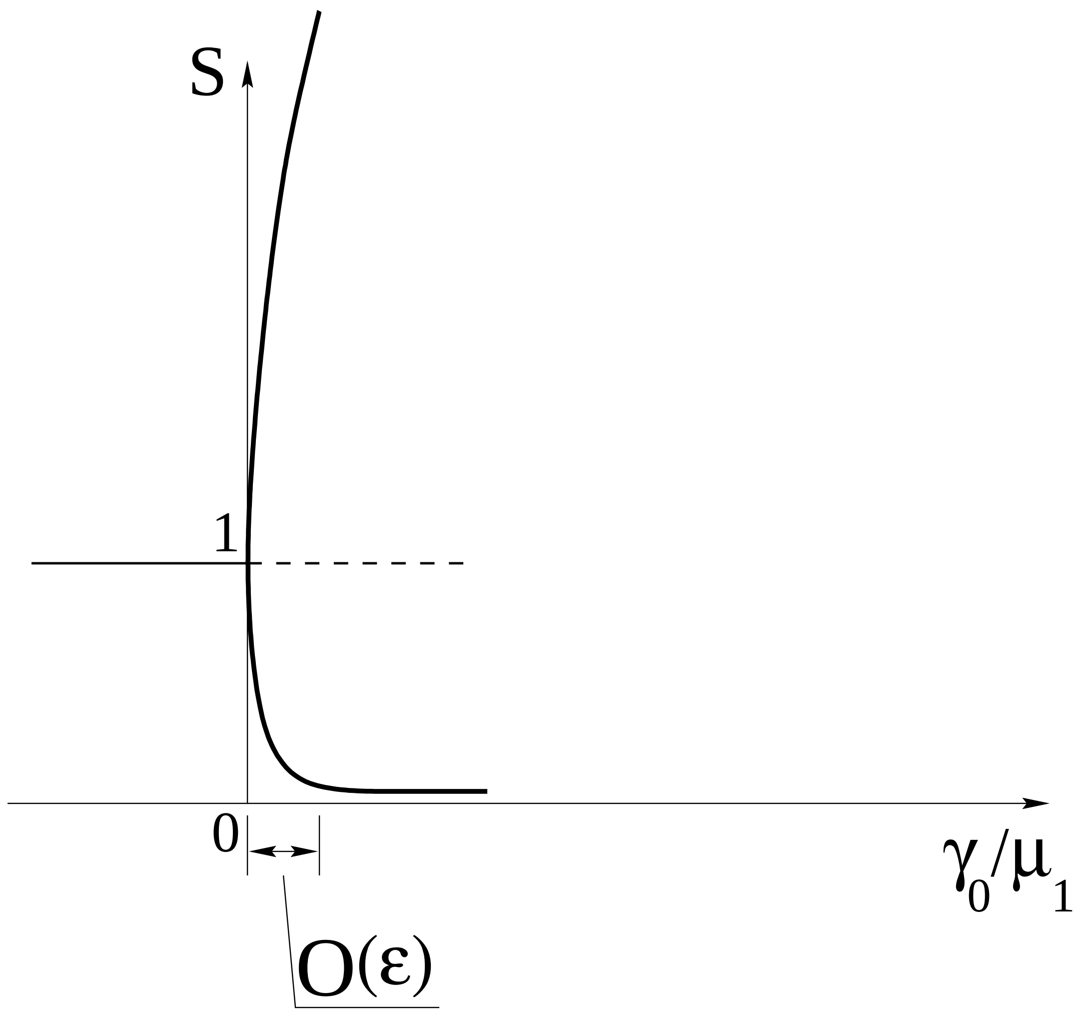


Figure 4: The size of the limit cycle must increase sharply as γ_0 crosses zero.

3 The case of vector N

Next, let us show that the situation is not essentially better when we consider the case of multidimensional N. Namely, let $N \in R^k$ and let $\mathcal{N}^\circ$ be a smooth $(k-1)$-dimensional piece of the threshold set on which exactly one eigenvalue of $H(N)$ has zero real part. We denote this eigenvalue as $\lambda(N)$ and assume that $\mathrm{Re}\lambda(N)$ changes with non-zero velocity when N crosses $\mathcal{N}^\circ$.

The equation near the threshold will still have the form (16) although N is not a scalar now. We introduce the coordinates near $\mathcal{N}^\circ$ such that $N = (N^0, n)$ where $N^0 \in R^{k-1}$ is the projection to the surface $\mathcal{N}^\circ$, and n plays, thus, the role of the distance to $\mathcal{N}^\circ$. So, we have a system of the kind

$$\begin{aligned}
\dot{S} &= 2(\mathrm{Re}\lambda(N) - \mathrm{Re}\;\alpha(N)\dot{n} - \mathrm{Re}\;\alpha^0(N)\dot{N}^0)S,\\
\dot{n} &= \varepsilon(f(N) - (g(N) + \varepsilon\tilde{g}(N,\sqrt{S}))S),\\
\dot{N}^0 &= \varepsilon(f^0(N) - g^0(N)S - \varepsilon\tilde{g}^0(N,\sqrt{S})S).
\end{aligned} \tag{24}$$

The condition that $\mathrm{Re}\lambda$ changes with non-zero velocity across the threshold means that we may assume

$$\mathrm{Re}\lambda(N) = n\beta(N) \tag{25}$$

with some function $\beta \neq 0$. For more definiteness we assume that

$$\beta(N) > 0 \tag{26}$$

everywhere near $\mathcal{N}^\circ$. As above, we assume $g(N) > 0$ and $f(N) > 0$. Like in the previous case ($N \in R^1$) it can be shown that this is the only reasonable choice for the signs of f and g. So this will be our standing assumption.

Recall that we must stay in a small neighborhood of the threshold, so the value of n must be small. Note that if we change the variable n as follows: $n \mapsto n_{new}(\Psi_0(N^0) + n\Psi_1(N^0))$, the system will not change its form, just the functions $f, g, \tilde{g}$ in the equation for $\dot{n}$ will change:

$$f_{new}(N) = f(N)\frac{(1-n\Psi_1(N^0))^2}{\Psi_0(N^0)} - n\left[\frac{\Psi_0'(N^0)}{\Psi_0(N^0)}(1-n\Psi_1(N^0)) + n\Psi_1'(N^0)\right]f^0(N),$$

$$g_{new}(N) = g(N)\frac{(1-n\Psi_1(N^0))^2}{\Psi_0(N^0)} - n\left[\frac{\Psi_0'(N^0)}{\Psi_0(N^0)}(1-n\Psi_1(N^0)) + n\Psi_1'(N^0)\right]g^0(N),$$

$$\tilde{g}_{new}(N,\sqrt{S}) = \tilde{g}(N,\sqrt{S})\frac{(1-n\Psi_1(N^0))^2}{\Psi_0(N^0)} - n\left[\frac{\Psi_0'(N^0)}{\Psi_0(N^0)}(1-n\Psi_1(N^0)) + n\Psi_1'(N^0)\right]\tilde{g}^0(N,\sqrt{S}),$$

as well as the functions α, α^0 and β in the equation for $\dot{S}$:

$$\alpha_{new}(N) = \frac{\Psi_0(N^0)}{(1-n\Psi_1(N^0))^2}\alpha(N),$$

$$\alpha^0_{new}(N) = \alpha^0(N) + \frac{n\Psi_0(N^0)}{(1-n\Psi_1(N^0))^2}\alpha(N^0)\left[\frac{\Psi_0'(N^0)}{\Psi_0(N^0)}(1-n\Psi_1(N^0)) + n\Psi_1'(N^0)\right],$$

$$\beta_{new}(N) = \beta(N)\frac{\Psi_0(N^0)}{1-n\Psi_1(N^0)}.$$

It is seen that we can always choose the scaling factors $\Psi_0(N^0)$ and $\Psi_1(N^0)$ such that the new functions β and f will satisfy the relation

$$\beta(N) = f(N) + O(n^2) \tag{27}$$

for small n, so we will assume that this relation holds indeed.

Like we did it in the case of scalar N, let us change the variables

$$S \mapsto S_{new} f(N)/(g(N) + \varepsilon\tilde{g}(N,\sqrt{S})). \tag{28}$$

The system takes the form

$$\begin{aligned}
\dot{S} &= (2n\beta(N)(1-\varepsilon\hat{g}(N,S)) - \tilde{\alpha}(N)\dot{n} - \tilde{\alpha}^0(N)\dot{N}^0 + O(\varepsilon^2))S,\\
\dot{n} &= \varepsilon f(N)(1-S),\\
\dot{N}^0 &= \varepsilon(f^0(N) - \frac{f(N)}{g(N)}g^0(N)S + O(\varepsilon))
\end{aligned} \tag{29}$$

where $\tilde{\alpha}(N) = 2\mathrm{Re}\ \alpha(N) + \frac{\partial}{\partial n}(\ln f(N)/g(N))$, $\tilde{\alpha}^0(N) = 2\mathrm{Re}\ \alpha^0(N) + \frac{\partial}{\partial N^0}(\ln f(N)/g(N))$ and $\hat{g}(N,S) = \frac{S}{g(N)}\frac{d}{dS}\tilde{g}(N,\sqrt{Sf(N)/g(N)})$.

We will also scale the time to $f(N)$ and write the system as

$$\begin{aligned}
\dot{S} &= (2n\mu(N)(1-\varepsilon\hat{g}(N,S)) - \gamma(N)\dot{n} - \gamma^0(N^0)\dot{n}^0 + O(\varepsilon^2))S,\\
\dot{n} &= \varepsilon(1-S),\\
\dot{N}^0 &= \varepsilon(F^0(N) - G^0(N)S + O(\varepsilon))
\end{aligned} \tag{30}$$

where $\mu(N) = \beta(N)/f(N)$, $\gamma(N) = \tilde{\alpha}(N)/f(N)$, $\gamma^0(N) = \tilde{\alpha}^0(N)/f(N)$, $F^0(N) = f^0(N)/f(N)$, $G^0(N) = g^0(N)/g(N)$. Note that $\mu(N) = 1 + O(n^2)$ according to (27).

We will now take explicitly into account that n must be small (as it is the distance to the threshold). Thus, we must scale $n \mapsto \delta n_{new}$ for some appropriate small δ. As in the previous case of the scalar N, we choose $\delta = \sqrt{\varepsilon}$.

Let us expand

$$\mu(N) = 1 + \mu_2(N^0)\varepsilon n^2 + \ldots,$$

$$\gamma(N) = \gamma_0(N^0) + \gamma_1(N^0)\sqrt{\varepsilon}n + \ldots,$$

$$\gamma^0(N) = \gamma_0^0(N^0) + \gamma_1^0(N^0)\sqrt{\varepsilon}n + \dots,$$
$$F^0(N) = F^0(N^0) + F^1(N^0)\sqrt{\varepsilon}n + \dots,$$
$$G^0(N) = G^0(N^0) + G^1(N^0)\sqrt{\varepsilon}n + \dots.$$

We also rescale time to the factor $\sqrt{\varepsilon}$. Thus, the system assumes the form (compare it with (21)):

$$\begin{aligned} \dot{S} &= (2n - \sqrt{\varepsilon}(\tilde{\gamma}_0(N^0)(1-S) + \zeta(N^0)) + \varepsilon(\varphi(N^0, n) + n\psi(N^0, S)) + O(\varepsilon^{3/2}))S, \\ \dot{n} &= 1 - S, \\ \dot{N}^0 &= \sqrt{\varepsilon}(F^0(N^0) - G^0(N^0)S) + \varepsilon(F^1(N^0) - G^1(N^0)S)n + O(\varepsilon^{3/2}) \end{aligned} \tag{31}$$

where

$$\tilde{\gamma}_0 = \gamma_0 + \gamma_0^0 G^0,$$

and ζ, φ, ψ are some smooth functions.

One more change of variables, namely $N^0 \mapsto N^0_{new} + \sqrt{\varepsilon}G^0(N^0)n$ and $n \mapsto n_{new} + \frac{1}{2}\sqrt{\varepsilon}\zeta(N^0)$ brings, finally, the system to the form

$$\begin{aligned} \dot{S} &= (2n - \sqrt{\varepsilon}\tilde{\gamma}_0(N^0)(1-S) + \varepsilon(\varphi(N^0, n) + n\psi(N^0, S)) + O(\varepsilon^{3/2}))S, \\ \dot{n} &= 1 - S + \varepsilon(\tilde{F}(N^0) - \tilde{G}(N^0)S) + O(\varepsilon^{3/2}), \\ \dot{N}^0 &= \sqrt{\varepsilon}(F^0(N^0) - G^0(N^0)) + \varepsilon(\tilde{F}^0(N^0) - \tilde{G}^0(N^0)S)n + O(\varepsilon^{3/2}) \end{aligned} \tag{32}$$

with some smooth $\tilde{F}$, $\tilde{G}$, $\tilde{F}^0$, $\tilde{G}^0$, φ and ψ.

At $\varepsilon = 0$ this system takes the form

$$\begin{aligned} \dot{S} &= 2nS, \\ \dot{n} &= 1 - S, \\ \dot{N}^0 &= 0. \end{aligned} \tag{33}$$

It possesses first integrals: N^0 and $h = S - \ln S + n^2$. Thus, the behavior of system (32) can be described, in general terms, as a rotation in the (S, n)-plane transverse to the threshold, governed by a slow evolution of h and N^0. To understand this evolution we will average the system with respect to the fast rotation. Namely, we consider the following truncated system

$$\begin{aligned} \dot{S} &= (2n - \sqrt{\varepsilon}\tilde{\gamma}_0(N^0)(1-S))S, \\ \dot{n} &= 1 - S, \\ \dot{N}^0 &= \sqrt{\varepsilon}(F^0(N^0) - G^0(N^0)). \end{aligned} \tag{34}$$

Note that the evolution of h in the full system (32) is governed by an equation

$$\dot{h} = \sqrt{\varepsilon}\tilde{\gamma}_0(N^0)(1-S)^2 + \varepsilon(\varphi(N^0,n)(S-1) + n\tilde{\psi}(N^0,S)) + O(\varepsilon^{3/2}) \tag{35}$$

for some smooth $\tilde{\psi}$, while in the truncated system we have

$$\dot{h} = \sqrt{\varepsilon}\tilde{\gamma}_0(N^0)(1-S)^2. \tag{36}$$

Let us take any point (S, n, N^0). In the conservative system (33) a periodic orbit $(S^*(t), n^*(t), N^0 = const)$ passes through this point, corresponding to the constant level line of N^0 and h (i.e. $S^*(t) - \ln S^*(t) + n^*(t)^2 = h = const$). Let $T(h)$ be the period of this orbit (if we choose an equilibrium of (33) as an initial point, i.e. if $h = 0$ and $S^*(t) \equiv 1$, $n^*(t) \equiv 0$, we take $T(0) = \lim_{h\to 0} T(h) = \pi\sqrt{2}$). It is obvious that the finite time t shift by the full system (33) deviates from that in the truncated system on the value of order $O(\varepsilon)$. Moreover, the deviation of the slow variables N^0 and h for the time t is estimated as follows:

$$\begin{aligned} \Delta N^0 &= \varepsilon \textstyle\int_0^t (\tilde{F}^0(N^0) - \tilde{G}^0(N^0)S^*(t))n^*(t)dt + O(\varepsilon^{3/2}), \\ \Delta h &= \varepsilon(\textstyle\int_0^t \varphi(N^0, n^*(t))(S^*(t)-1)dt + \int_0^t \tilde{\psi}(N^0, S^*(t))n^*(t)dt) + O(\varepsilon^{3/2}). \end{aligned} \tag{37}$$

Since $n^*(t)dt = d\ln S^*(t)/2$ and $(S^*(t)-1)dt = -dn^*(t)$ (recall that (S^*, n^*) is a trajectory of the conservative system (33)), it follows that the integrals in (37) vanish at the moment of time $t = T(h)$ (i.e. when t equals to the period of $(S^*(t), n^*(t))$). Thus, at $t = T(h)$, the deviation between the values of N^0 and h in the full system and in the truncated system is of order $O(\varepsilon^{3/2})$ only. Since $\dot{N}^0$ and $\dot{h}$ are small (of order $O(\sqrt{\varepsilon})$), it now immediately follows that
the values of N^0 and h in the full system stay $O(\varepsilon)$-close to those in the truncated system for the times of order $O(\varepsilon^{-1/2})$.

If we scale time to $\varepsilon^{-1/2}$, then we will have the $O(\varepsilon)$-closeness to the system

$$\begin{aligned} \dot{S} &= (2\varepsilon^{-1/2}n - \tilde{\gamma}_0(N^0)(1-S))S, \\ \dot{n} &= \varepsilon^{-1/2}(1-S), \\ \dot{N}^0 &= F^0(N^0) - G^0(N^0) \end{aligned} \tag{38}$$

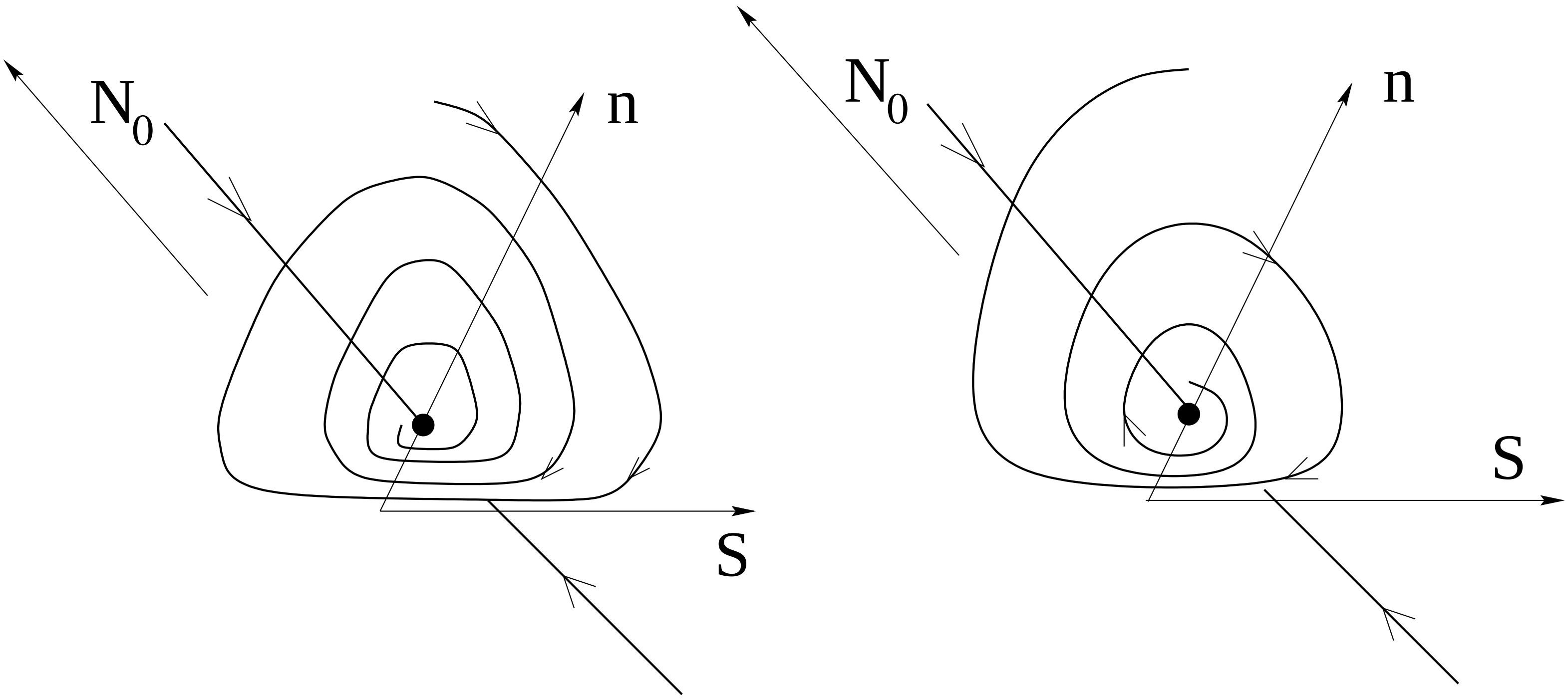
for finite times. It follows immediately, that if the system

$$\dot{N}^0 = F^0(N^0) - G^0(N^0) \tag{39}$$

has an attractor (e.g. a stable equilibrium state or a stable limit cycle), the value of N^0 for the full system will stay in a small neighborhood of the attractor forever (in an $O(\varepsilon)$-neighborhood in the case of an exponentially stable attractor).

In the simplest case when this attractor is a stable equilibrium state N^*, the behavior is the same as in the case of scalar N. Indeed, we have

$$\dot{h} = \tilde{\gamma}_0(N^0)(1-S)^2 \tag{40}$$



$\tilde{\gamma}_0(N^*) < 0$ $\tilde{\gamma}_0(N^*) > 0$

Figure 5: Phase portraits for N^0 close to the equilibrium state of the averaged system (39).

in the system (38), so if $\tilde{\gamma}_0(N^*) < 0$, then all the orbits of (38) must tend to the equilibrium $(S = 1, n = 0, N^0 = N^*)$, and if $\tilde{\gamma}_0(N^*) > 0$, we have that all the orbits of (38) tend to infinity (except for those which lie in the stable manifold $(S = 1, n = 0)$ of the now saddle equilibrium $(S = 1, n = 0, N^0 = N^*)$. For the full system (32) we have that if $\tilde{\gamma}_0(N^*) < 0$, then any orbit comes into an $O(\varepsilon)$-neighborhood of the point $(S = 1, n = 0, N^0 = N^*)$, and one can indeed show that the full system has a stable equilibrium state which attracts all the orbits in this neighborhood – hence it is attractive globally. If $\tilde{\gamma}_0(N^*) > 0$ we have that all the orbits leave the region of finite h, except for those in the stable manifold of a saddle equilibrium state, $O(\varepsilon)$-close to $(S = 1, n = 0, N^0 = N^*)$ (Fig.5). So, like in the case of scalar N we have that we could possibly observe self-pulsations only in the region of parameters for which $\tilde{\gamma}_0(N^*) = O(\varepsilon)$.

In the case where the attractor of (39) is a stable limit cycle $L = \{N^0 = N^*(t)\}_{t\in[0,\tau]}$, the behavior of h in the system (38) averaged with respect to fast oscillations in (S, n)-variables is governed by the equation

$$\dot{h} = \tilde{\gamma}_0(N^0)\frac{1}{T(h)}\int_0^{T(h)}(1 - S^*(t))^2 dt + O(\sqrt{\varepsilon}). \tag{41}$$

So, if we introduce

$$\bar{\gamma} = \int_0^\tau \tilde{\gamma}_0(N^*(t))dt$$

where τ is the period of the slow limit cycle L, then at $\bar{\gamma} < 0$ all the orbits of (38) tend to L on the manifold $(S = 1, n = 0)$, and at $\bar{\gamma} > 0$ all the orbits of (38) tend to infinity (except for those which lie in the stable manifold $(S = 1, n = 0)$ of L). For the full system (32) we have that if $\bar{\gamma} < 0$, then any orbit come into an $O(\varepsilon)$-neighborhood of L on the manifold $(S = 1, n = 0)$, and one can show that the full system has a stable limit cycle which attracts all the orbits in this neighborhood. If $\bar{\gamma} > 0$ we have that all the orbits leave the region of finite h, except for those in the stable manifold of a saddle limit cycle $O(\varepsilon)$-close to L.

Thus, we can have stable self-pulsations in this case, provided the system (39), which describes the averaged behavior in the projection to the threshold, has a stable limit cycle and the corresponding value of $\bar{\gamma}$ is negative. Note that the oscillations in the S-variable seem to be small here: $S = 1 + O(\varepsilon)$. Recall, however, that we have scaled the variable S to a factor depending on N^0 (see (28)), so finite-amplitude oscillations of optical power are indeed present in this regime (see Fig.6): in the original variables we have

$$|E(t)|^2 = |\xi(N^*(t))|^2 f(N^*(t))/g(N^*(t)) + O(\varepsilon).$$

The main disadvantage here is that the frequency of such self-pulsations is low: it is $O(\sqrt{\varepsilon})$ times lower than the frequency of oscillations transverse to the threshold. As above, we can possible have nontrivial fast regimes in this case only in a thin parameter region where $\bar{\gamma} = O(\varepsilon)$.

4 Non-transverse threshold crossing

Better results are obtained when we drop the condition, that the critical eigenvalue of $H(N)$ in (4) crosses the imaginary axis with a non-zero velocity. To see the effect, we assume again that $N \in R^1$. We assume that the matrix $H(N)$ depends smoothly on some parameter c varying near zero, and $H(N)$ has, at $c = 0$, exactly one eigenvalue $\lambda(N)$ on the threshold at some $N = N^0$ such that

$$\mathrm{Re}\lambda'(N^0) = 0 \tag{42}$$

(i.e. the non-degeneracy condition (15) is now broken). So, we have our system in the form (16) where we can assume, according to (42), that

$$\mathrm{Re}\lambda(N) = ac + b(N - N^0)^2 + o((N - N^0)^2) \tag{43}$$

with some a and b which are generically non-zero (note that all terms in this formula, including a, b and N^0, are now functions of the parameter c, though it is not

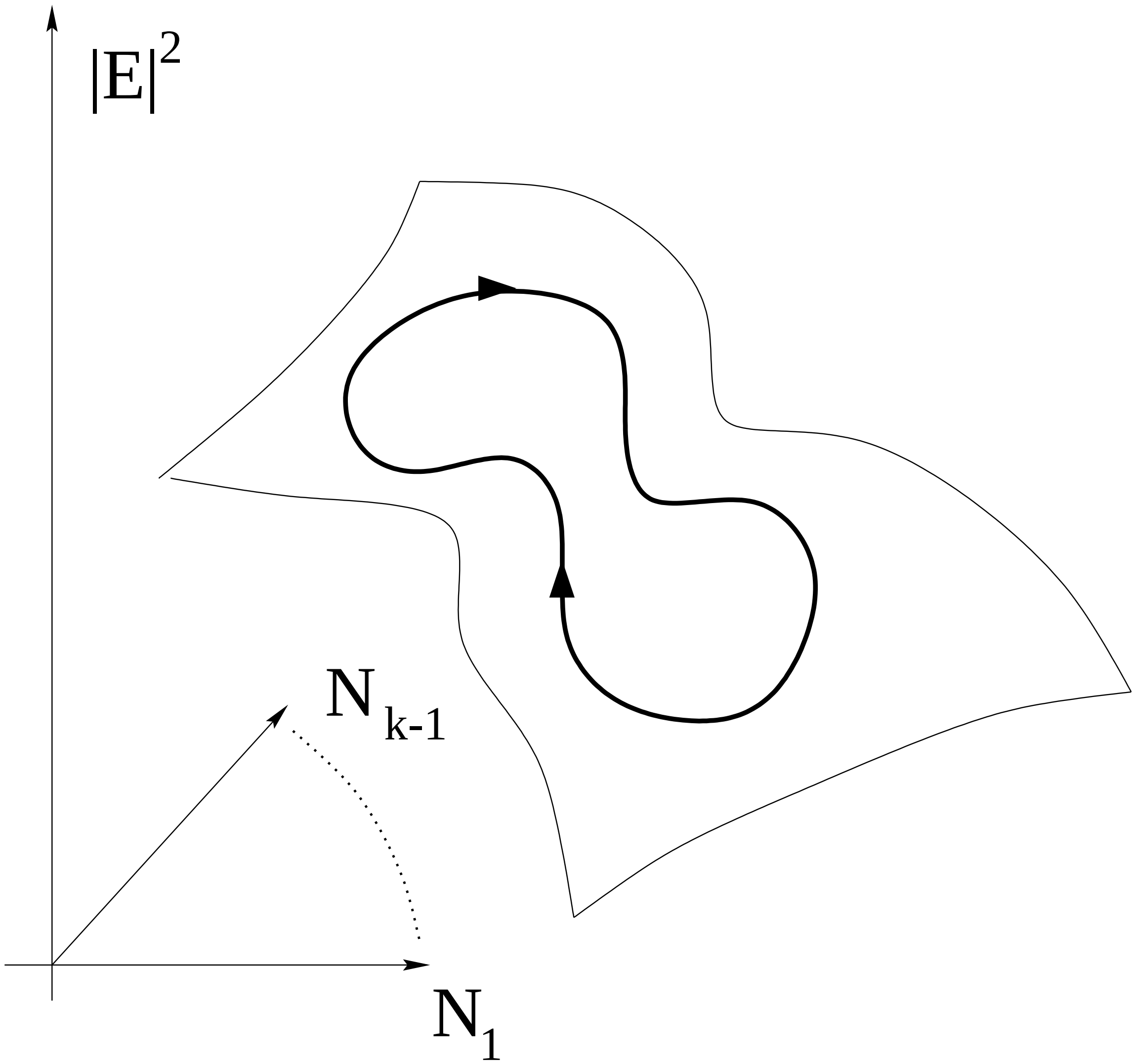


Figure 6: The limit cycle in the averaged system (39) produces slow oscillations of optical power which passively follow the oscillations in N^0.

important for the sequel). As above, the system (16) can be brought to the form (see (18))

$$\dot{S} = (2\mu(N)(1 - \varepsilon\hat{g}(N,S)) - \gamma(N)\dot{N} + O(\varepsilon^2))S,$$
$$\dot{N} = \varepsilon(1 - S) \tag{44}$$

where, at $c = 0$, the function $\mu(N)$ vanishes at $n = N^0$ along with the first derivative. By scaling the parameter c if necessary, we may write $\mu(N)$ as follows

$$\mu(N) = \frac{1}{2}\mu_2(-c + (N - N^0)^2) + O((N - N^0)^3), \tag{45}$$

with some non-zero coefficient μ_2. By scaling N and time both to $|\mu_2|^{-1/3}$ we can always achieve

$$|\mu_2| = 1, \tag{46}$$

so this will be our standing assumption.

Let us choose the scaling parameter $\delta = \varepsilon^{1/3}$, so we will write

$$N = N^0 + \varepsilon^{1/3}n$$

and expand

$$\mu(N) = \mu_2(-c + \varepsilon^{2/3})n^2 + \varepsilon\tilde{\mu}(n,\varepsilon)$$

and

$$\gamma(N) = \gamma_0 + \gamma_1\varepsilon^{1/3}n + O(\varepsilon^{2/3}).$$

The system takes the form

$$\dot{S} = \varepsilon^{2/3}(\mu_2(-C + n^2) + \varepsilon^{1/3}\tilde{\mu}(n,\varepsilon) + \varepsilon^{1/3}(\gamma_0 + \varepsilon^{1/3}\gamma_1 n)(1 - S) + O(\varepsilon))S,$$
$$\dot{n} = \varepsilon^{2/3}(1 - S)$$

where $C = c\varepsilon^{-2/3}$ can take now arbitrary finite values (recall that c and ε are small parameters).

After rescaling the time to the factor $\varepsilon^{2/3}$, the system takes the form

$$\dot{S} = (\mu_2(-C + n^2) + \varepsilon^{1/3}\tilde{\mu}(n,\varepsilon) + \varepsilon^{1/3}(\gamma_0 + \varepsilon^{1/3}\gamma_1 n)(1 - S) + O(\varepsilon))S,$$
$$\dot{n} = 1 - S. \tag{47}$$

This system is $O(\varepsilon^{1/3})$-close to the conservative system

$$\dot{S} = (\mu_2(-C + n^2) + \varepsilon^{1/3}\tilde{\mu}(n,\varepsilon))S,$$
$$\dot{n} = 1 - S. \tag{48}$$

The latter has a first integral

$$h = \mu_2(-Cn + \frac{1}{3}n^3) + \varepsilon^{1/3}\int\tilde{\mu}(n,\varepsilon))dn + S - \ln S \tag{49}$$

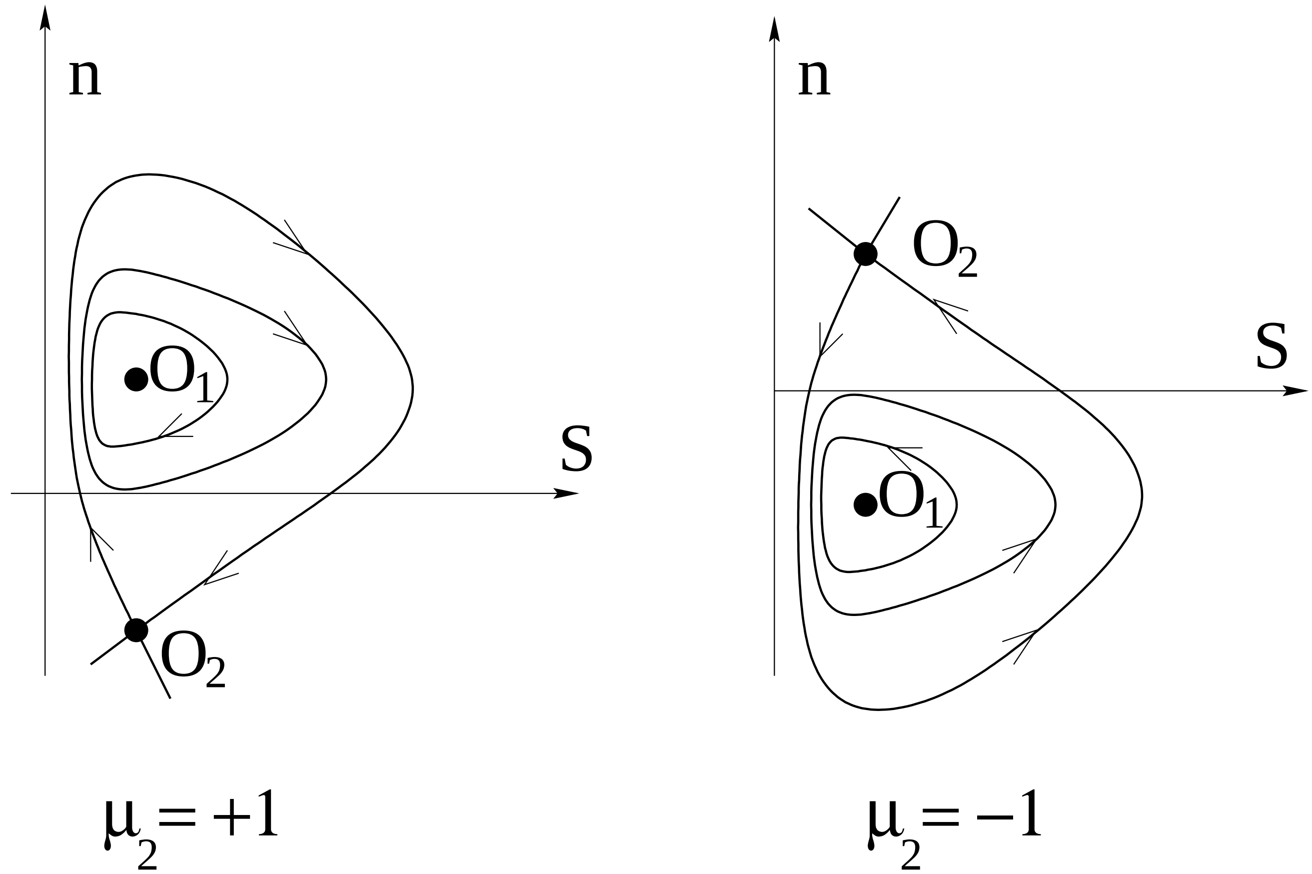


Figure 7: Phase portraits for the integrable limit (48).

which has two critical points: $O_1(n = \mu_2\sqrt{C}+O(\varepsilon^{1/3}), S = 1)$ and $O_2(n = -\mu_2\sqrt{C}+O(\varepsilon^{1/3}), S = 1)$ at $C > 0$ (recall that $\mu_2 = \pm 1$). These points are the equilibria of system (48), O_1 is a center (h has minimum at O_1) and O_2 is a saddle (Fig.7). The values of h between $h(O_1) = 1 - \frac{2}{3}\sqrt{C^3} + O(\varepsilon^{1/3})$ and $h(O_2) = 1 + \frac{2}{3}\sqrt{C^3} + O(\varepsilon^{1/3})$ correspond to periodic orbits of system (48) surrounding O_1. We will denote such an orbit as $(S^*(t,h), n^*(t,h))$ (assuming that $t = 0$ corresponds to the intersection of the orbit with a segment of the straight line $S = 1$ between O_1 and O_2) and its period will be denoted as $T^*(h)$.

In the full system (47) we have two equilibria as well, close to the equilibria of the conservative system (48). We denote them as O_1 and O_2, respectively. The latter is, of course, a saddle at small ε (because it is a saddle at $\varepsilon = 0$). To determine the stability of O_1, let us compute $\dot{h}$ along the trajectories of the full system:

$$\dot{h} = -\varepsilon^{1/3}(\gamma_0 + \varepsilon^{1/3}\gamma_1 n)(1 - S)^2 + O(\varepsilon)). \tag{50}$$

It is seen that h decays at $\gamma_0 > 0$ and grows at $\gamma_0 < 0$ (if ε is sufficiently small). Thus, O_1 is stable at $\gamma_0 > 0$ and unstable at $\gamma_0 < 0$. Moreover, we see that if γ_0 is bounded away from zero there cannot be periodic orbits in system (47) at small ε.

The stability loss of O_1 when γ_0 decreases across zero must be accompanied by the Andronov-Hopf bifurcation, so limit cycles must exist at small γ_0. To estimate the parameter region corresponding to the existence of the limit cycles, we do the following. First, we assume that $\gamma_1 \neq 0$ in (47). Since we are interested in the region of small γ_0, we can introduce a rescaled parameter

$$\Gamma = \frac{\gamma_0}{\gamma_1}\varepsilon^{-1/3}.$$

The system is rewritten as

$$\begin{aligned} \dot{S} &= (\mu_2(-C+n^2) + \varepsilon^{1/3}\tilde{\mu}(n,\varepsilon) + \varepsilon^{2/3}\gamma_1(\Gamma+n)(1-S) + O(\varepsilon))S, \\ \dot{n} &= 1 - S. \end{aligned} \tag{51}$$

Take a segment of the straight line $S = 1$ between the points O_1 and O_2 as a cross-section. The points on the cross-section are parametrized by the values of h ranging from $h(O_1)$ to $h(O_2)$. It is obvious that the orbit $(S(t,h), n(t,h))$ of the full system, starting at $t = 0$ on the cross-section, is estimated as

$$S(t) = S^*(t) + O(\varepsilon^{1/3}), \qquad n(t) = n^*(t) + O(\varepsilon^{1/3})$$

for finite times t. It follows, that the return time of the orbit to the cross-section (the segment of the line $S = 1$) is estimated as

$$T(h) = T^*(h) + O(\varepsilon^{1/3}).$$

Thus, the new value of h at the moment when the orbit returns to the cross-section (see (50)) is given by

$$\bar{h} = h + \int_0^{T(h)} \dot{h}dt = h - \varepsilon^{2/3}\gamma_1 \int_0^{T^*(h)} (\Gamma + n^*(t,h))(1 - S^*(t,h))^2 dt + O(\varepsilon). \tag{52}$$

Formula (52) defines the Poincaré map $h \mapsto \bar{h}$ on the cross-section. The fixed points of this map correspond to limit cycles. According to (52), we have a limit cycle $L(h)$ corresponding to the fixed point at a given value of h when

$$\Gamma = -\frac{\oint_{L_h^*} nSdn}{\oint_{L_h^*} Sdn} + O(\varepsilon^{1/3}) \tag{53}$$

where L_h^* is the closed curve $(S = S^*(t,h), n = n^*(t,h))$, i.e. it is the closed integral curve of the conservative system (48), corresponding to the given value of h (when proceeding from (52) to (53) we used that $(1 - S^*)dt = dn$). Thus, for every $h \in (h(O_1), h(O_2))$ we have a unique value of the parameter Γ for which the full system (47) has a limit cycle $L(h)$. The limit cycle shrinks to the equilibrium state O_1 as $h \to h(O_1)$ and it merges into a homoclinic loop to O_2 as $h \to h(O_2)$). Thus, on the (Γ, C)-plane we have bifurcational curves, corresponding to the Andronov-Hopf bifurcation and to the bifurcation of a homoclinic loop; both curves are given by the equation (53) where one should put $h = h(O_1)$ and $h = h(O_2)$, respectively.

Thus, the Andronov-Hopf bifurcation curve is given by the equation

$$\Gamma = -\mu_2\sqrt{C} + O(\varepsilon^{1/3}).$$

Routine computations show that the first Lyapunov value does not vanish on this curve for every finite C and small ε. Therefore, only one limit cycle is born when crossing this curve. It is also not hard to check that we have $|\Gamma| < \sqrt{C}$ (at ε small) on the bifurcational curve which corresponds to the separatrix loop. This means that this curve does not intersect the Andronov-Hopf bifurcation curve and that the saddle value (the sum of characteristic exponents at the saddle) does not vanish. The latter means, again, that only one limit cycle is born at the homoclinic bifurcation.

Note that both bifurcational curves start at the point $(C, \Gamma) = 0 + O(\varepsilon^{1/3})$ which corresponds to Bogdanov-Takens bifurcation of an equilibrium with double zero characteristic exponent (see Fig.8).

The limit cycle $L(h)$ is stable when $\frac{d\bar{h}}{dh} < 1$, i.e. at

$$\gamma_1 \left\{ \Gamma \frac{d}{dh} \oint_{L_h} S dn + \frac{d}{dh} \oint_{L_h} nS dn \right\} < 0$$

(when ε is small enough). By (53), the condition of stability of $L(h)$ can be written as

$$\gamma_1 \left\{ \oint_{L_h^*} nS dn \frac{d}{dh} \oint_{L_h^*} S dn - \oint_{L_h^*} S dn \frac{d}{dh} \oint_{L_h^*} nS dn \right\} < 0$$

(recall that $\oint_{L_h} S dn = \oint_{L_h} (S-1) dn = -\int_0^{T^*(h)} (1 - S^*(t,h))^2 dt < 0$). Further, we can rewrite it as

$$\gamma_1 \left\{ 2 \int_0^{T_h^*} (1 - S^*)^2 n^* dt \int_0^{T_h^*} (1 - S^*) \frac{\partial S^*}{\partial h} dt \right.$$

$$\left. - \int_0^{T_h^*} (1 - S^*)^2 dt \int_0^{T_h^*} (2(1 - S^*) n^* \frac{\partial S^*}{\partial h} - (1 - S^*)^2 \frac{\partial n^*}{\partial h}) dt \right\} > 0.$$

Now note that

$$\frac{\partial n^*(t,h)}{\partial h} = (1 - S^*(t,h)) \int_{t_0}^{t} \frac{S^*(s,h)}{(1 - S^*(s,h))^2} ds$$

for some irrelevant $t_0(h)$ and

$$\frac{\partial S^*(t,h)}{\partial h} = -\frac{d}{dt} \frac{\partial n^*(t,h)}{\partial h}$$

(check that these solve the variational equations for the conservative system (48)). From these formulas it is easy to compute (by integration by parts) that

$$2 \int_0^{T_h^*} (1 - S^*) \frac{\partial S^*}{\partial h} dt = -T^*(h)$$

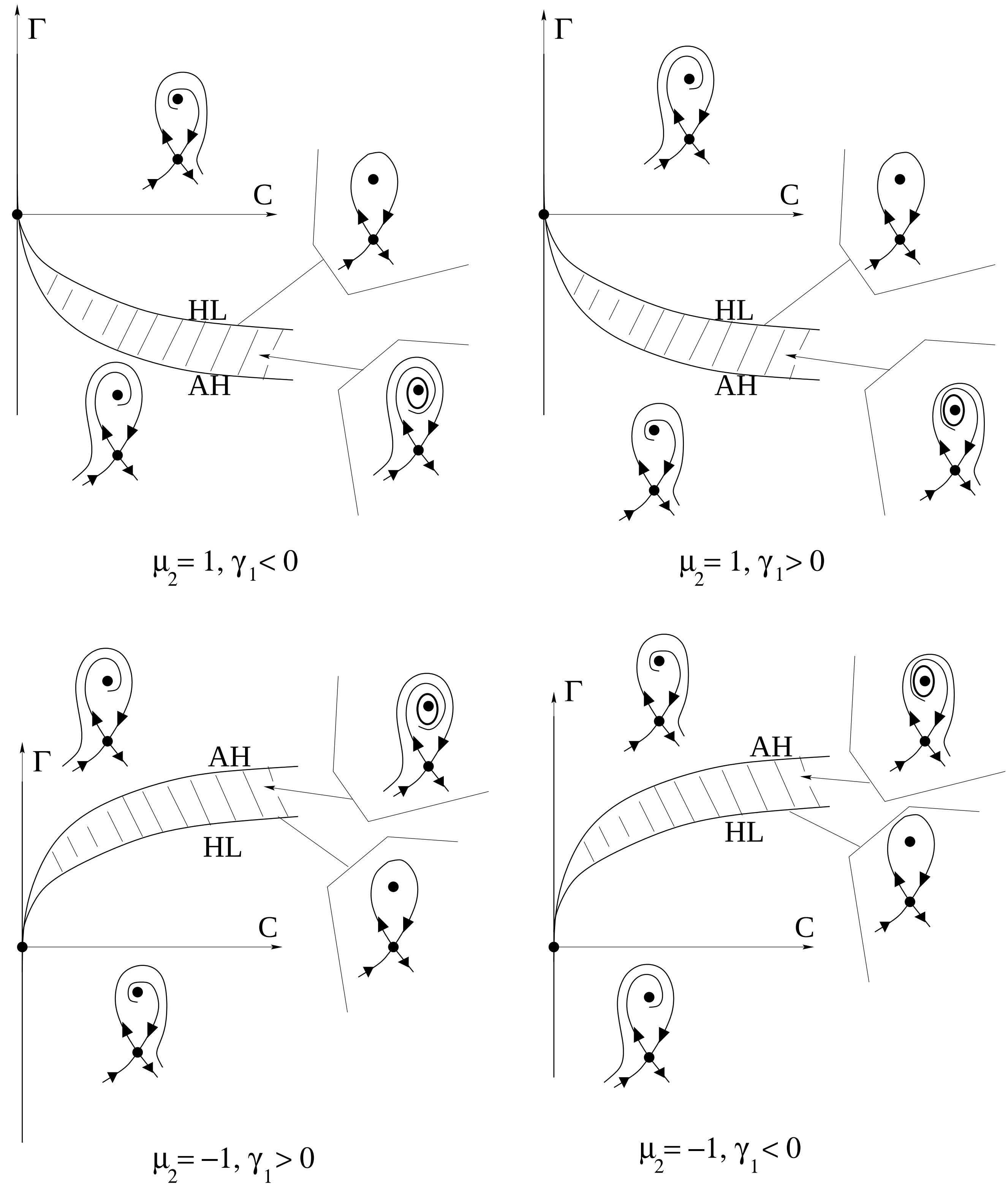


Figure 8: Bifurcation diagrams for the system (51) at small ε. Not more than one limit cycle may exist here.

and

$$\int_0^{T_h^*} (2(1-S^*)n^* \frac{\partial S^*}{\partial h} - (1-S^*)^2 \frac{\partial n^*}{\partial h}) dt = -\int_0^{T^*(h)} n^*(t,h)dt.$$

Thus, if we denote

$$\bar{n}(h) = \frac{1}{T^*(h)} \int_0^{T^*(h)} n^*(t,h)dt,$$

the condition of stability of the limit cycle $L(h)$ may finally be written as

$$\gamma_1 \left\{ \oint_{L_h^*} Sndn - \bar{n}(h) \int_{L_h^*} Sdn \right\} > 0. \tag{54}$$

As numerical evidence shows, the expression in the figure brackets is always negative at $\mu_2 = +1$ and positive at $\mu_2 = -1$. Therefore, everywhere in the region between the Andronov-Hopf curve and the homoclinic loop bifurcation curve on the (C, Γ)-plane there exists a unique limit cycle, which is stable at $\gamma_1\mu_2 < 0$ and it is unstable at $\gamma_1\mu_2 > 0$.

So, if $\gamma_1\mu_2 < 0$, we can always have a non-empty interval of values of h corresponding to the stable limit cycle. Hence, for any finite value of C (i.e. for values of the original non-rescaled parameter c of order $O(\varepsilon^{2/3})$) we have a finite interval of values of Γ, corresponding to a stable limit cycle. Thus, in the case under consideration, the parameter values corresponding to self-pulsations occupy a region of size $\sim$ $(\text{const} \cdot \varepsilon^{2/3}) \times (\text{const} \cdot \varepsilon^{1/3})$ on the plane $(c, \gamma_0/\gamma_1)$, which is, of course, better than the $O(\varepsilon)$-size region in the previous case.

Note that when proceeding from the original system (44) to its rescaled form (47) we scaled time to the factor $\varepsilon^{2/3}$. Therefore, the frequency of the obtained limit cycle becomes of order $O(\varepsilon^{2/3})$ as we return to the original variables (note also that this frequency tends to zero as the limit cycle approaches the homoclinic loop). So, the frequency in this case is lower than that in the case of transverse threshold crossing, considered in section 2. Indeed, the time-scaling factor (when we proceeded from the original model (16) to the rescaled model (21)) was there proportional to $\varepsilon^{1/2}$, and this is the factor which gives the asymptotics for the frequency of the limit cycle which could appear there at the sharp Andronov-Hopf bifurcation.

Let us briefly discuss the case of cubic degeneracy in $\text{Re}\lambda(N)$ at the threshold crossing, i.e. we assume now

$$\text{Re}\lambda'(N^0) = 0, \qquad \text{Re}\lambda''(N^0) = 0. \tag{55}$$

In this case we will consider, as a model, the following system:

$$\begin{aligned} \dot{S} &= (\mu_0 + \mu_1(N-N^0) + \mu_3(N-N^0)^3 - (\gamma_0 + \gamma_1(N-N^0))\dot{N})S, \\ \dot{N} &= \varepsilon(1-S) \end{aligned} \tag{56}$$

where $\mu_3 = \pm 1$, and μ_0, μ_1 are small parameters which unfold the cubic degeneracy. As above, to ensure the existence of a limit cycle, the parameter γ_0 has also to be taken small, while γ_1 will be taken nonzero. We scale $N - N^0$ to $\varepsilon^{1/4}$ and time to $\varepsilon^{-3/4}$. Equations take the form

$$
\begin{aligned}
\dot{S} &= (C_0 + C_1 n + \mu_3 n^3 - \gamma_1 \varepsilon^{1/2}(\Gamma + n)\dot{n})S, \\
\dot{n} &= 1 - S
\end{aligned}
\tag{57}
$$

where $C_0 = \mu_0/\varepsilon^{3/4}$, $C_1 = \mu_1/\varepsilon^{1/2}$ and $\Gamma = \varepsilon^{-1/4}\gamma_0/\gamma_1$ are rescaled parameters, $n = (N - N^0)/\varepsilon^{1/4}$ is the scaled distance to the threshold.

At $\varepsilon = 0$ this system has a first integral

$$
h = \frac{\mu_3}{4} n^4 + C_0 n + C_1 \frac{n^2}{2} + S - \ln S. \tag{58}
$$

In the case $\mu_3 > 0$, constant levels of h are composed of closed curves. If L_h^* is such a curve corresponding to a given value h, then a limit cycle is born from L_h^* if $\lim\limits_{\varepsilon \to 0} \frac{1}{\varepsilon^{1/2}} \oint_{L_h^*} \dot{h} dt = 0$, where $\dot{h}$ is the derivative of h with respect to the system (57). This gives (compare it with (53) that the limit cycle is born from L_h^* at small ε if

$$
\Gamma = -\frac{\oint_{L_h^*} nS dn}{\oint_{L_h^*} S dn}. \tag{59}
$$

Variation of h in a finite interval corresponds to a finite range of values of Γ in this formula, so we have a finite range of values of parameter Γ corresponding to the existence of the limit cycle in system (57), and this is true for arbitrary finite values of C_0 and C_1. The same formula (59) is valid in the case $\mu_3 < 0$, as well; one should note, however, that the closed curves L_h^* exist here only for a bounded range of values of h and for $|C_0| < 2|\mu_3|(C_1/3|\mu_3|)^{3/2}, C_1 > 0$. In any case, we have the existence of limit cycles for finite regions in the space of parameters (C_0, C_1, Γ). Returning to the original, non-rescaled parameters (μ_0, μ_1, γ_0) we obtain that the region of existence of self-pulsations in the model (57) has the size $O(\varepsilon^{3/4}) \times O(\varepsilon^{1/2}) \times O(\varepsilon^{1/4})$. The frequency of the oscillations is of order $O(\varepsilon^{3/4})$.

Looking once again over Secs.2,4 (see (21),(51),(57)) one can propose some general form for a rescaled single-mode laser model with scalar N:

$$
\begin{aligned}
\dot{S} &= (\Phi(n) - \delta(\Gamma + n)\dot{n})S, \\
\dot{n} &= 1 - S,
\end{aligned}
\tag{60}
$$

where Φ is some polynomial with arbitrary coefficients, Γ is a finite parameter and δ is a small parameter (some fractional power of the original small parameter ε). This system is δ-close to a conservative one with the first integral

$$
h = \int \Phi(n) dn + S - \ln S.
$$

As above, a closed curve L_h^* of the conservative system which corresponds to a given value of h produces a limit cycle at non-zero δ if (59) is satisfied. Note that the case of a transverse threshold crossing can also be modeled by a system of the same form (see (21)). However, $\Phi(n)$ must be linear here, and in this case we have

$$\oint_{L_h^*} nSdn \sim \oint_{L_h^*} \dot{S}dn = \oint_{L_h^*} \dot{n}dS \equiv 0$$

which, according to (59) gives zero interval of values of Γ for which limit cycles can appear. This unfortunate identity is basically the main reason why we have an anomalously small region of the existence of self-pulsations in the case of transverse threshold crossing, so it is that fundamental obstacle to self-pulsations which is mentioned in the title of this paper.

5 Degenerate mode on the threshold

As we saw in the previous Sections, if only a single electromagnetic mode is excited, then system (14) which governs laser dynamics becomes close to conservative after an appropriate rescaling of time and N-variables. Essentially non-conservative dynamics appears if a double mode comes to the threshold, i.e. if the matrix $H(N)$ in (1) (hence matrix $A(N)$ in (14)) has a double eigenvalue with zero real part.

In this case $U \in C^2$ in (14) and the matrix A is a Jordan block

$$\begin{pmatrix} i\omega_0 & 1 \\ 0 & i\omega_0 \end{pmatrix} \tag{61}$$

with some real nonzero ω_0. Note that matrices reducible to this form compose a codimension-3 surface in the 8-dimensional space of complex (2×2)-matrices (indeed, they must satisfy three real equalities: the real part of one of the eigenvalues equals to the real part of the other one and equals to zero, and the imaginary parts of the eigenvalues are equal). Therefore, such configuration of eigenvalues can generically appear only in three-parameter families of matrices. We assume here that N is scalar, so we have a one-parameter family of matrices $A(N)$. Thus, to study the bifurcation of a double mode on the threshold we must assume that our system (14) depends on two independent real parameters (the choice of parameters will be specified later).

Recall that arbitrary linear transformations (with N-dependent coefficients) of the variables U do not change the form of equations (14). Therefore, we will apply such transformations in order to make the matrix A as simple as possible. It is easy to see (see also [14]) that any matrix close to (61) can be brought, by a linear transformation depending smoothly on the coefficients of the matrix, to the following form

$$A(N) = \begin{pmatrix} i\omega(N) & 1 \\ \mu(N) + i\delta(N) & i\omega(N) - \lambda(N) \end{pmatrix} \tag{62}$$

where $\mu(N)$, $\delta(N)$, $\lambda(N)$ are real and close to zero, and the real quantity $\omega(N)$ is close to ω_0. At the critical moment, when $A(N)$ is given by (61), we have μ, δ, and

λ vanished. We assume that

$$\mu'(N) \neq 0 \tag{63}$$

at the critical moment. Therefore, by implicit function theorem, for any system which is close to the given one, there exists a value N^0 such that $\mu(N^0) = 0$. We take the corresponding values of $\delta(N^0) \equiv \delta_0$ and $\lambda(N^0) \equiv \lambda_0$ as independent small parameters which govern the bifurcations. By changing $N \mapsto N - N^0$ we can always assume $N^0 = 0$. By (63) we can also assume

$$\mu(N) \equiv N \tag{64}$$

(this would require a smooth change of the coordinate N, which do not change the form of equations (14), obviously). We can also expand

$$\delta(N) = \delta_0 + \delta_1 N + \delta_2 N^2 + \ldots, \qquad \lambda(N) = \lambda_0 + \lambda_1 N + \ldots. \tag{65}$$

Equations (14) can be written now as

$$\begin{aligned}
\dot{U}_1 &= i\omega(N)U_1 + U_2 + O(\varepsilon),\\
\dot{U}_2 &= (N + i\delta(N))U_1 + (i\omega(N) - \lambda(N))U_2 + O(\varepsilon),\\
\dot{N} &= \varepsilon(F(N) - |U_1|^2 G(N) + O(|U_2| \cdot \|U\| + O(\varepsilon)).
\end{aligned} \tag{66}$$

Recall that this system must be invariant with respect to the phase rotation: $(U_1, U_2) \mapsto (U_1, U_2)e^{i\varphi}$. Therefore, by choosing a rotation coordinate frame, we can always make $\omega(N)$ identically zero, without changing other coefficients of the equations.

The further analysis shows that stable stationary states may exist here only if $F(0) > 0$ and $G(0) > 0$, so we will make this sign assumption.

Assume now that $\delta_1 \neq 0$. Let us make rescaling:

$$U_1 \mapsto u\sqrt{F(0)/G(0)}, N \mapsto \tau^2 n$$

where $\tau^3 = \varepsilon F(0)$. Then, scaling the time to τ^{-1} we arrive to the following system

$$\begin{aligned}
\ddot{u} &= (n + i(D_0 + D_1 n))u - L\dot{u} + O(\varepsilon^{1/3}),\\
\dot{n} &= 1 - |u|^2 + O(\varepsilon^{1/3})
\end{aligned}$$

where

$$D_0 = \frac{\delta_0}{(\varepsilon F(0))^{2/3}}, \qquad L = \frac{\lambda_0}{(\varepsilon F(0))^{1/3}}; \tag{67}$$

D_0 and L are rescaled parameters which can take arbitrary finite values, and we denote $D_1 = \delta_1$ for uniformity.

These equations are $\varepsilon^{1/3}$-close to

$$\begin{aligned}
\ddot{u} &= (n + i(D_0 + D_1 n))u - L\dot{u},\\
\dot{n} &= 1 - |u|^2.
\end{aligned} \tag{68}$$

This system is not conservative (we take $L > 0$ to assure dissipation), so it may have attractors. Indeed, let $u = re^{i\varphi}$. Then the system recasts as

$$\ddot{r} + L\dot{r} = (\Omega^2 + n)r,$$
$$\dot{\Omega} + \Omega(L + 2\dot{r}/r) = D_0 + D_1 n, \tag{69}$$
$$\dot{n} = 1 - r^2$$

where we denote $\Omega = \dot{\varphi}$. Equilibria of this system give stationary states of (68), limit cycles in (69) correspond to periodic self-pulsations.

At $L^2 + 4D_0D_1 > 0$ system (69) has two equilibria:

$$O_1 : (r = 1, \Omega = \Omega_1 = \frac{-L + \sqrt{L^2 + 4D_0D_1}}{2D_1}, n = -\Omega_1^2),$$

and

$$O_2 : (r = 1, \Omega = \Omega_2 = \frac{-L - \sqrt{L^2 + 4D_0D_1}}{2D_1}, n = -\Omega_1^2).$$

O_2 is always saddle, O_1 is stable when

$$L^2 + 4\Omega^2 > 2L\sqrt{L^2 + 4D_0D_1} + \frac{1}{L}.$$

On the boundary of this region O_1 undergoes a non-degenerate Andronov-Hopf bifurcation, which means that we have in the plane of parameters (L, D_0) a finite region of existence of a stable limit cycle.

By (67), for sufficiently small ε, in the plane of the original non-rescaled parameters (λ_0, δ_0) we have the $O(\varepsilon^{1/3}) \times O(\varepsilon^{2/3})$-size region of existence of stable self-pulsations in system (66). The time rescaling factor τ, when proceeding from (66) to (68), was of order $\varepsilon^{1/3}$, so this is the order of the frequency of the self-pulsations we have found.

Note that the dynamics of (69) is, of course, richer then just a simple periodic behavior. As numerics shows, the limit cycle born at the Andronov-Hopf bifurcation may lose its stability and a chaotic regime may appear after a chain of, say, period-doubling bifurcations. The attractor dies via bifurcations of homoclinic loops when it collides with the saddle equilibrium state O_2, and the orbits seemingly escape to infinity.

There is a similarity between this case and the case of quadratic tangency to the imaginary axis considered in the previous Section. In both cases we have two stationary states, one of which changes stability and this gives rise to self-pulsations, whereas the other stationary state is a saddle which bounds the attraction domain. Also, in both cases we have the same estimate for the size of the region of existence of self-pulsations. However, the frequency of self-pulsations in the present case is relatively higher than in the case of the non-transverse threshold crossing: $O(\varepsilon^{1/3})$ vs. $O(\varepsilon^{2/3})$.

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